A note on the moving hyperplane method

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Let us consider the problem:

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u \in C^1(\overline{\Omega}), & u > 0 \text{ in } \Omega \end{cases} \tag{1}$$

where $1 < p \leq 2$, $\Omega \subset \mathbb{R}^N$ is a bounded convex domain, Δ_p is the p-Laplacian operator defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and $f : \mathbb{R} \to [0, +\infty)$ is continuous on $\mathbb{R}$, locally Lipschitz continuous on $(0, +\infty)$ and satisfies

$$\exists C_0, C_1 > 0 \text{ such that } C_0|u|^q \leq f(u) \leq C_1|u|^q \quad \forall u \in \mathbb{R}^+$$

where $q > p - 1$. In [1], Ph. Clément and the first author proved the existence of a nontrivial positive solution to (1) by using continuation methods and establishing a priori estimates for the solutions of some nonlinear eigenvalue problem associated with (1). The desired a priori estimates use a blow up argument as well as some monotonicity and symmetry results proved by Damascelli and Pacella in [3] and generalizing to the p-Laplacian operator with $1 < p < 2$ the well known results of Gidas–Ni–Nirenberg from [4] and Berestycki–Nirenberg in [2]. In their proof, Damascelli and Pacella use a new technique consisting in moving hyperplanes orthogonal to directions close to a fixed one. To be efficient, this procedure needs some continuity of some parameters linked with the moving plane method (see the functions $\lambda_1(\nu)$ and $a(\nu)$ defined below). Therefore they assume in their result that $\partial\Omega$ is smooth to insure this continuity (and only for that reason). However, such a smoothness hypothesis does not appear in the case $p = 2$ in the classical moving plane procedure (see [2]).
Our purpose here is to give more precision on the regularity of the domain Ω that is needed to have the continuity of the function $a(\nu)$ and the lower semicontinuity of $\lambda_1(\nu)$, and so to have the monotonicity and symmetry results of [3]. This question is also important concerning the existence result from [1]. Specifically, we ask that the domain be of class C^1, and we also discuss convexity conditions relating to the continuity of $\lambda_1(\nu)$.

In this paper, Ω will denote an open bounded domain in $\mathbb{R}^N$ with C^1 boundary. We will say that Ω is strictly convex if for all $x, y \in \overline{\Omega}$ and for all $t \in (0, 1)$, $(1 - t)x + ty \in \Omega$.
For any direction $\nu \in \mathbb{R}^N$, $|\nu| = 1$, we define

$$a(\nu) := \inf_{x \in \Omega} x.\nu$$

and for all $\lambda \geq a(\nu)$,

$$\Omega_\lambda^\nu := \{x \in \Omega \,|\, x.\nu < \lambda\}, \qquad T_\lambda^\nu := \{x \in \Omega \,|\, x.\nu = \lambda\} \,(\neq \emptyset \text{ for } a(\nu) < \lambda < -a(-\nu)).$$

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Let us denote by R_λ^ν the symmetry with respect to the hyperplane T_λ^ν and

$$\begin{array}{rcl} x_\lambda^\nu & := & R_\lambda^\nu(x) \ \forall x \in \mathbb{R}^N, \\ (\Omega_\lambda^\nu)' & := & R_\lambda^\nu(\Omega_\lambda^\nu), \\ \Lambda_1(\nu) & := & \{\mu > a(\nu) \,|\, \forall \lambda \in (a(\nu), \mu), \text{ we have (2) and (3)}\}, \\ \lambda_1(\nu) & := & \sup \Lambda_1(\nu) \end{array}$$

where (2), (3) are the following conditions:

$$(\Omega_\lambda^\nu)' \text{ is not internally tangent to } \partial\Omega \text{ at some point } p \notin T_\lambda^\nu \tag{2}$$

$$\text{for all } x \in \partial\Omega \cap T_\lambda^\nu,\ \nu(x).\nu \neq 0, \tag{3}$$

where $\nu(x)$ denotes the inward unit normal to $\partial\Omega$ at x. Notice that $\Lambda_1(\nu) \neq \emptyset$ and $\lambda_1(\nu) < \infty$ since for $\lambda > a(\nu)$ close to $a(\nu)$, (2) and (3) are satisfied and Ω is bounded.

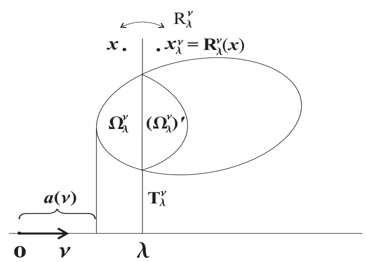

Figure 1: Illustration of the notations

Propositions 1 and 2 below give sufficient conditions on Ω to guarantee the continuity of the functions $a(\nu)$ and $\lambda_1(\nu)$, as well as the lower semicontinuity of $\lambda_1(\nu)$.

Proposition 1 *Let Ω be a bounded domain with C^1 boundary. Then the function $a(\nu)$ is continuous with respect to $\nu \in S^{N-1}$.*

Proposition 2 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^1 boundary. Then the function $\lambda_1(\nu)$ is lower semicontinuous with respect to $\nu \in S^{N-1}$. If moreover Ω is strictly convex, then $\lambda_1(\nu)$ is continuous.*

As a consequence of these results, we can give more precision on the conditions to impose to Ω in the monotonicity result of [3]. This result becomes:

Theorem (Damascelli-Pacella, Theorem 1.1 from [3]) *Let Ω be a bounded domain in $\mathbb{R}^N$ with C^1 boundary, $N \geq 2$ and $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function. Let $u \in C^1(\bar{\Omega})$ be a weak solution of*

$$\begin{cases} -\Delta_p u = g(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $1 < p < 2$. Then, for any direction $\nu \in \mathbb{R}^N$ and for λ in the interval $(a(\nu), \lambda_1(\nu)]$, we have $u(x) \leq u\,(x_\lambda^\nu)$ for all $x \in \Omega_\lambda^\nu$. Moreover $\frac{\partial u}{\partial \nu}(x) > 0$ for all $x \in \Omega_{\lambda_1(\nu)}^\nu \backslash Z$ where $Z = \{x \in \Omega \,|\, \nabla u(x) = 0\}$.

Below we prove Propositions 1 and 2 and we give a counterexample of a C^∞ convex but not strictly convex domain for which $\lambda_1(\nu)$ is not continuous everywhere.

Proof of Proposition 1: Let us fix a direction $\nu \in S^{N-1}$. We shall prove that for all sequence $\nu_n \to \nu$ with $|\nu_n| = 1$, there exists a subsequence still denoted by ν_n such that $a(\nu_n) \to a(\nu)$. Since Ω is bounded, $(a(\nu_n))$ is also bounded, so passing to an adequate subsequence, there exists $\bar{a} \in \mathbb{R}$

such that $a(\nu_n) \to \bar{a}$. We will show that $\bar{a} = a(\nu)$. Suppose by contradiction that $\bar{a} \neq a(\nu)$. Then either $\bar{a} < a(\nu)$ or $\bar{a} > a(\nu)$.

<u>CASE 1: $\bar{a} < a(\nu)$:</u> Since

$$a(\nu) = \inf_{x\in\Omega} x.\nu = \min_{x\in\overline{\Omega}} x.\nu = \min_{x\in\partial\Omega} x.\nu,$$

there exists $x_n \in \partial\Omega$ such that

$$x_n.\nu_n = a(\nu_n). \tag{4}$$

Passing again to a subsequence, there exists $x \in \partial\Omega$ such that $x_n \to x$ and taking the limit of (4), we get $x.\nu = \bar{a} < a(\nu)$, a contradiction with the definition of $a(\nu)$.

<u>CASE 2: $\bar{a} > a(\nu)$:</u> There exists $x \in \partial\Omega$ with $x.\nu = a(\nu)$. For n large, $|x.\nu_n - x.\nu| = |x.\nu_n - a(\nu)|$ is small, and since $a(\nu_n) \to \bar{a} > a(\nu)$, for n large enough we have $x.\nu_n < a(\nu_n)$, contradicting the definition of $a(\nu_n)$. ■

Proof of Proposition 2: We first prove the continuity of $\lambda_1(\nu)$ if Ω is strictly convex. Suppose by contradiction that there exists $\nu \in S^{N-1}$ such that λ_1 is not continuous at ν. Then we can fix $\varepsilon > 0$ and a sequence $(\nu_n) \subset S^{N-1}$ such that $\nu_n \to \nu$ and $|\lambda_1(\nu) - \lambda_1(\nu_n)| > \varepsilon$ for all $n \in \mathbb{N}$. Passing to a subsequence still denoted by (ν_n), we can suppose that

$$\text{either} \quad \lambda_1(\nu) > \lambda_1(\nu_n) + \varepsilon \quad \forall n \in \mathbb{N} \quad \text{or} \quad \lambda_1(\nu) < \lambda_1(\nu_n) - \varepsilon \quad \forall n \in \mathbb{N}.$$

<u>CASE 1: $\lambda_1(\nu) > \lambda_1(\nu_n) + \varepsilon$ for all $n \in \mathbb{N}$.</u> For any fixed $n \in \mathbb{N}$, we have the following alternative: either there exists $x_n \in T^{\nu_n}_{\lambda_1(\nu_n)} \cap \partial\Omega$ with $\nu(x_n).\nu_n = 0$, or there exists $x_n \in \left(\partial\Omega \cap \overline{\Omega^{\nu_n}_{\lambda_1(\nu_n)}}\right) \setminus T^{\nu_n}_{\lambda_1(\nu_n)}$ with $(x_n)^{\nu_n}_{\lambda_1(\nu_n)} \in \partial\Omega$. Passing once again to subsequences, we can suppose that we are in one of the two situations above for all $n \in \mathbb{N}$. We treat below each situation and try to reach a contradiction.

(1.a) For all $n \in \mathbb{N}$, there exists $x_n \in T^{\nu_n}_{\lambda_1(\nu_n)} \cap \partial\Omega$ with $\nu(x_n).\nu_n = 0$.

Passing if necessary to a subsequence, there exist $\bar{\lambda} \leq \lambda_1(\nu) - \varepsilon$ and $x \in T^{\nu}_{\bar{\lambda}} \cap \partial\Omega$ such that $x_n \to x$ and $\nu(x).\nu = 0$. This contradicts the definition of $\lambda_1(\nu)$.

(1.b) For all $n \in \mathbb{N}$, there exists $x_n \in \left(\partial\Omega \cap \overline{\Omega^{\nu_n}_{\lambda_1(\nu_n)}}\right) \setminus T^{\nu_n}_{\lambda_1(\nu_n)}$ with $(x_n)^{\nu_n}_{\lambda_1(\nu_n)} \in \partial\Omega$.

Passing if necessary to a subsequence, there exist $\bar{\lambda} \leq \lambda_1(\nu) - \varepsilon$ and $x \in \partial\Omega \cap \overline{\Omega^{\nu}_{\bar{\lambda}}}$ such that $x_n \to x$ and $x^{\nu}_{\bar{\lambda}} \in \partial\Omega$. If $x \notin T^{\nu}_{\bar{\lambda}}$, we reach a contradiction with the definition of $\lambda_1(\nu)$. Suppose now that $x \in T^{\nu}_{\bar{\lambda}}$. Let us denote $(x_n)^{\nu_n}_{\lambda_1(\nu_n)}$ by u_n. Since Ω is a C^1 domain, it holds that $\nu(u_n).\nu_n \leq 0$ for all n. By definition of $\lambda_1(\nu_n)$, $\nu(x_n).\nu_n \geq 0$. If $x \in T^{\nu}_{\bar{\lambda}}$, $x = \lim x_n = \lim u_n$ and so $\nu(x).\nu = 0$, which contradicts the definition of $\lambda_1(\nu)$.
Observe that we do not use the convexity of the domain in Case 1.

<u>CASE 2: $\lambda_1(\nu) < \lambda_1(\nu_n) - \varepsilon$ for all $n \in \mathbb{N}$.:</u> As in the first case, either there exists $x \in T^{\nu}_{\lambda_1(\nu)} \cap \partial\Omega$ with $\nu(x).\nu = 0$ or there exists $x \in \left(\partial\Omega \cap \overline{\Omega^{\nu}_{\lambda_1(\nu)}}\right) \setminus T^{\nu}_{\lambda_1(\nu)}$ such that $x^{\nu}_{\lambda_1(\nu)} \in \partial\Omega$. We treat the first situation in (2.a) and the second one in (2.b).

(2.a) For ε small enough, $T^{\nu}_{\lambda_1(\nu)+\frac{\varepsilon}{2}} \cap \partial\Omega \neq \emptyset$. Since Ω is strictly convex, there exists $x' \in T^{\nu}_{\lambda_1(\nu)+\frac{\varepsilon}{2}} \cap \partial\Omega$ such that

$$\nu(x').\nu < 0. \tag{5}$$

For $\varepsilon > 0$ small enough, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the sets $T^{\nu_n}_{\lambda_1(\nu)+\frac{\varepsilon}{2}} \cap \partial\Omega$ are non empty and since they are compact, we can choose a sequence (x_n) satisfying

$$x_n \in T^{\nu_n}_{\lambda_1(\nu)+\frac{\varepsilon}{2}} \cap \partial\Omega, \qquad |x' - x_n| = \min\left\{|x' - y| \,:\, y \in T^{\nu_n}_{\lambda_1(\nu)+\frac{\varepsilon}{2}} \cap \partial\Omega\right\}.$$

Passing if necessary to a subsequence, $x_n \to y$ for some $y \in T^{\nu}_{\lambda_1(\nu)+\frac{\varepsilon}{2}} \cap \partial\Omega$ such that

$$|x' - y| = \lim_{n\to\infty} \text{dist}\left(x', T^{\nu_n}_{\lambda_1(\nu)+\frac{\varepsilon}{2}} \cap \partial\Omega\right),$$

but since this limit is equal to 0, we infer that $x' = y$. Now, since $\lambda_1(\nu) < \lambda_1(\nu_n) - \varepsilon$ for all $n \in \mathbb{N}$, $\nu(x_n).\nu_n > 0$ for all n and thus $\nu(x').\nu \geq 0$, a contradiction with (5).

(2.b) The convexity of Ω implies that $x^{\nu}_{\lambda_1(\nu)+\frac{\varepsilon}{2}} \notin \overline{\Omega}$. Now, $x^{\nu_n}_{\lambda_1(\nu)+\frac{\varepsilon}{2}} \to x^{\nu}_{\lambda_1(\nu)+\frac{\varepsilon}{2}}$, so that

$$x^{\nu_n}_{\lambda_1(\nu)+\frac{\varepsilon}{2}} \notin \overline{\Omega} \tag{6}$$

for n large enough. But since $x.\nu < \lambda_1(\nu)$ by definition of x, we also have $x.\nu_n < \lambda_1(\nu) < \lambda_1(\nu) + \frac{\varepsilon}{2}$ for n sufficiently large, and so

$$x \in \left(\partial\Omega \cap \Omega^{\nu_n}_{\lambda_1(\nu)+\frac{\varepsilon}{2}}\right) \setminus T^{\nu_n}_{\lambda_1(\nu)+\frac{\varepsilon}{2}}$$

for these values of n. This fact together with (6) contradicts the definition of $\lambda_1(\nu_n)$.

The proof of the lower semicontinuity follows from Case 1, which uses only the C^1 regularity of the domain. ■

A counterexample in $\mathbb{R}^2$

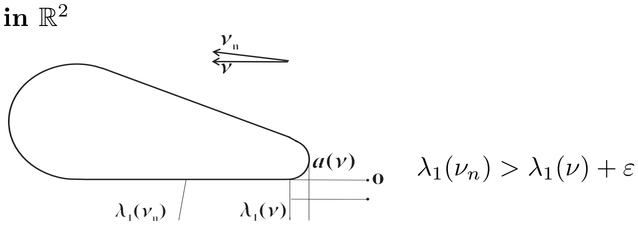

Figure 2: Counterexample of a smooth convex but not strictly convex domain for which $\lambda_1(\nu)$ is not continuous everywhere.

This is an example of a convex but not strictly convex domain in $\mathbb{R}^2$. It contradicts case (2.a) in the proof and indeed, case (2.a) is the only one using the *strict* convexity. The example can be made smooth. In fact all is required is a convex domain in $\mathbb{R}^2$ whose boundary contains a piece of (straight) line, say of length L. Then for ν parallel to the line, there exists a sequence $\nu_n \to \nu$ such that $\lambda_1(\nu_n) \geq \lambda_1(\nu) + \frac{L}{2}$.

A variation of this construction will produce similar examples in higher dimensions.

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